Theoretical quantum optics

Summer semester 2018

Worksheet 1

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The aim of this worksheet is to analyze Einstein's pre-quantum theory of absorption and emission of light by atoms and study its application to an elementary model of the laser. The articles discuss double-slit experiments with single electrons and molecules.

Exercise 1: Einstein's theory of atom-radiation interaction

In 1917, Einstein formulated a phenomenological theory for spontaneous and stimulated emission and absorption.
Assume a closed cavity with N identical atoms with two relevant energy levels, E_a and E_b, quasi-resonant with the thermal radiation produced by the cavity walls at temperature T: $\hbar\omega = E_a - E_b$, where ω is the frequency of the photon. The average energy density of the thermal radiation is $u_T(\omega)$. Let A be the probability per unit time to spontaneously decay from level a to b, emitting a photon of energy $\hbar\omega$. On the other hand, if the atom is in state b, there will be a probability per unit time for absorption proportional to the energy present in the cavity; that is the absorption rate is $Bu_T(\omega)$.

1) Denoting by N_a and N_b the populations of the two levels ($N_a + N_b = N$), write the rate equations for $\dot{N}_a$ and $\dot{N}_b$, where "·ïs the time derivative.
2) Consider thermal equilibrium and assume that levels a and b are Boltzmann distributed. Determine $u_T(\omega)$.
3) Compare $u_T(\omega)$ with Planck's black body energy distribution and show that the latter is not recovered.
4) Einstein introduced a third process, called stimulated emission, with a rate given by $Bu_T(\omega)$. Write down the modified rate equation and show that now Planck's distribution is recovered. Determine A and B.
5) Solve the rate the rate equation from 4) with $N_a(0) = 0$. What is the lifetime of the upper level in the absence of thermal radiation?
6) We next assume that there is an additional external source of energy, e.g. a light beam crossing the cavity (like in a laser). The average density can then be written as $u(\omega) = u_T(\omega) + u_E(\omega)$, where E denotes the external energy source. Discuss the steady state properties of the solution of the rate equation. What is the maximum possible occupation of the upper level?

Exercise 2: Elementary theory of the laser

Let us consider two-level atoms resonant with the radiation field (and neglect for the time being spontaneous emission) as in Exercise 1. We write the rate equation in the form

$$\dot{N}_b = -\dot{N}_a = -WnN_b + WnN_a$$

where n is the number of photons in the cavity and W the transition rate from one level to the other. We define the population difference $D = N_a - N_b$ that obeys

$$\dot{D} = -2WnD - \frac{1}{T_1}(D - D_0)$$

The phenomenological second term on the right hand side accounts for spontaneous decay and pump action. T_1 is the characteristic lifetime associated with the decay of the population, while $D_0 = D(0)$ is the equilibrium population in the absence of photons. On the other hand, there is also a rate equation for the photons:

$$\dot{n} = WnD - \frac{n}{T_c}$$

where the last term describes the photons coming out of the cavity (T_c: lifetime of photons in the cavity).

1) *Threshold and population inversion.* Assuming initially a low photon number (say 1), amplification of the number of photons will only occur if $\dot{n} > 0$. Discuss the physical meaning of this condition: When is it satisfied? What are the implications for the level populations? How can the threshold be lowered?

2) *Steady state.* Determine and discuss the steady state solutions n_∞ and D_∞ corresponding to $\dot{D} = \dot{n} = 0$.

3) *Linear stability analysis.* Let us write the solutions of the laser rate equations in the form

   $$n(t) = n_\infty + \epsilon_n(t) \quad \text{and} \quad D(t) = D_\infty + \epsilon_D(t)$$

   where n_∞ and D_∞ are the steady state solutions and ϵ_n and ϵ_D are small deviations from the steady state.
   By linearizing the laser equations up to order ϵ, derive equations for $\dot{\epsilon}_n$ and $\dot{\epsilon}_D$. Using the ansatz

   $$\begin{pmatrix} \epsilon_n \\ \epsilon_D \end{pmatrix} = e^{\lambda t} \begin{pmatrix} \epsilon_n(0) \\ \epsilon_D(0) \end{pmatrix}$$

   discuss the stability ($\lambda < 0$) of the solutions for the steady state solutions found in 2). Plot a stability diagram for n_∞ as a function of D_0.

Exercise 3: Paper-Work

Find the following articles online and answer the following questions for each of them:

- What is the paper about?
- Why is it interesting?

- What is done?
- How is it done?

Demonstration of single electron buildup of an interference pattern
A. Tonomura, J. Endo, T. Matsuda, T. Kawasaki, and H. Ezawa,
American Journal of Physics **57**, 117 (1989)

Wave–particle duality of C_{60} molecules
Markus Arndt, Olaf Nairz, Julian Vos-Andreae, Claudia Keller, Gerbrand van der Zouw and Anton Zeilinger
Nature **401**, 680 (1999)

Observation of the Kapitza–Dirac effect
Daniel L. Freimund, Kayvan Aflatooni and Herman Batelaan
Nature **413**, 142 (2001)