The aim of this worksheet is to analyze Einstein’s pre-quantum theory of absorption and emission of light by atoms and study its application to an elementary model of the laser. The articles discuss double-slit experiments with single electrons and molecules.

**Exercise 1: Einstein’s theory of atom-radiation interaction**

In 1917, Einstein formulated a phenomenological theory for spontaneous and stimulated emission and absorption.

Assume a closed cavity with \( N \) identical atoms with two relevant energy levels, \( E_a \) and \( E_b \), quasi-resonant with the thermal radiation produced by the cavity walls at temperature \( T \): \( \hbar \omega = E_a - E_b \), where \( \omega \) is the frequency of the photon. The average energy density of the thermal radiation is \( u_T(\omega) \).

Let \( A \) be the probability per unit time to spontaneously decay from level \( a \) to \( b \), emitting a photon of energy \( \hbar \omega \). On the other hand, if the atom is in state \( b \), there will be a probability per unit time for absorption proportional to the energy present in the cavity; that is the absorption rate is \( B u_T(\omega) \).

1) Denoting by \( N_a \) and \( N_b \) the populations of the two levels \( (N_a + N_b = N) \), write the rate equations for \( \dot{N}_a \) and \( \dot{N}_b \), where \( \dot{\cdot} \) is the time derivative.

2) Consider thermal equilibrium and assume that levels \( a \) and \( b \) are Boltzmann distributed. Determine \( u_T(\omega) \).

3) Compare \( u_T(\omega) \) with Planck’s black body energy distribution and show that the latter is not recovered.

4) Einstein introduced a third process, called stimulated emission, with a rate given by \( B u_T(\omega) \). Write down the modified rate equation and show that now Planck’s distribution is recovered. Determine \( A \) and \( B \).

5) Solve the rate the rate equation from 4) with \( N_a(0) = 0 \). What is the lifetime of the upper level in the absence of thermal radiation?

6) We next assume that there is an additional external source of energy, e.g. a light beam crossing the cavity (like in a laser). The average density can then be written as \( u(\omega) = u_T(\omega) + u_E(\omega) \), where \( E \) denotes the external energy source. Discuss the steady state properties of the solution of the rate equation. What is the maximum possible occupation of the upper level?
Exercise 2: Elementary theory of the laser

Let us consider two-level atoms resonant with the radiation field (and neglect for the time being spontaneous emission) as in Exercise 1. We write the rate equation in the form

\[ \dot{N}_b = -\dot{N}_a = -WnN_b + WnN_a \]

where \( n \) is the number of photons in the cavity and \( W \) the transition rate from one level to the other. We define the population difference \( D = N_a - N_b \) that obeys

\[ \dot{D} = -2WnD - \frac{1}{T_1}(D - D_0) \]

The phenomenological second term on the right hand side accounts for spontaneous decay and pump action. \( T_1 \) is the characteristic lifetime associated with the decay of the population, while \( D_0 = D(0) \) is the equilibrium population in the absence of photons. On the other hand, there is also a rate equation for the photons:

\[ \dot{n} = WnD - \frac{n}{T_c} \]

where the last term describes the photons coming out of the cavity (\( T_c \): lifetime of photons in the cavity).

1) **Threshold and population inversion.** Assuming initially a low photon number (say 1), amplification of the number of photons will only occur if \( \dot{n} > 0 \). Discuss the physical meaning of this condition: When is it satisfied? What are the implications for the level populations? How can the threshold be lowered?

2) **Steady state.** Determine and discuss the steady state solutions \( n_\infty \) and \( D_\infty \) corresponding to \( \dot{D} = \dot{n} = 0 \).

3) **Linear stability analysis.** Let us write the solutions of the laser rate equations in the form

\[ n(t) = n_\infty + \epsilon_n(t) \quad \text{and} \quad D(t) = D_\infty + \epsilon_D(t) \]

where \( n_\infty \) and \( D_\infty \) are the steady state solutions and \( \epsilon_n \) and \( \epsilon_D \) are small deviations from the steady state.

By linearizing the laser equations up to order \( \epsilon \), derive equations for \( \dot{\epsilon}_n \) and \( \dot{\epsilon}_D \). Using the ansatz

\[ \begin{pmatrix} \epsilon_n \\ \epsilon_D \end{pmatrix} = e^{\lambda t} \begin{pmatrix} \epsilon_n(0) \\ \epsilon_D(0) \end{pmatrix} \]

discuss the stability (\( \lambda < 0 \)) of the solutions for the steady state solutions found in 2). Plot a stability diagram for \( n_\infty \) as a function of \( D_0 \).

Exercise 3: Paper-Work

Find the following articles online and answer the following questions for each of them:

- What is the paper about?
- Why is it interesting?
• What is done?

• How is it done?

*Demonstration of single electron buildup of an interference pattern*
A. Tonomura, J. Endo, T. Matsuda, T. Kawasaki, and H. Ezawa,

*Wave–particle duality of C\textsubscript{60} molecules*
Markus Arndt, Olaf Nairz, Julian Vos-Andreae, Claudia Keller, Gerbrand van der Zouw and Anton Zeilinger
Nature 401, 680 (1999)

*Observation of the Kapitza–Dirac effect*
Daniel L. Freimund, Kayvan Aflatooni and Herman Batelaan
Nature 413, 142 (2001)