Master equation for an arbitrarily quick driven harmonic oscillator

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Introduction

We derive a Redfield master equation for an arbitrarily quick driven harmonic oscillator that generates a CPTP-map due to the choice of the interaction Hamiltonian. Afterwards, we solve the master equation for arbitrary Gaussian initial states and calculate the mean energy and a coherence measure for Gaussian states from the solutions. Finally, we compare our results with an adiabatic master equation derived in [1] and with the O'Connell master equation. We find good agreement with the adiabatic master equation in the slow driving regime.

The master equation

Consider a driven harmonic oscillator with the Hamiltonian $H_S(t) = \hbar\omega(n+1/2) - \lambda(t)\,(a+a^\dagger)$ coupled to a bosonic bath with Hamiltonian $H_B = \sum_k \hbar\omega_k b_k^\dagger b_k$ through the interaction Hamiltonian

$$H_I = a^\dagger \otimes b + a \otimes b^\dagger \qquad (1)$$

with $b = \sum_k g_k b_k$. The bath is assumed to be initially in a thermal state $\rho_\beta = Z_\beta^{-1}\exp(-\beta H_B)$ with inverse temperature β and with the partition function $Z_\beta = \mathrm{Tr}_B(\exp(-\beta H_B))$. In the Born-Markov approximation the dynamics of the reduced density matrix in the interaction picture is governed by

$$\frac{d}{dt}\tilde{\rho}_S(t) = \int_0^t ds\,\big(C_{12}(s)\left[\tilde{a}(t-s)\tilde{\rho}_S(t), \tilde{a}^\dagger(t)\right] + C_{21}(s)\left[\tilde{a}^\dagger(t-s)\tilde{\rho}_S(t), \tilde{a}(t)\right]\big) + \text{H.c.} \qquad (2)$$

Hereby we have introduced the bath correlation functions $C_{ij}(s) = \langle \tilde{B}_i(s) B_j\rangle_\beta/\hbar^2$, where $\langle \dots \rangle_\beta = \mathrm{Tr}_B(\dots\rho_\beta)$ and with $B_1 = b$ and $B_2 = b^\dagger$. $C_{ij}(s)$ can be expressed by the ohmic spectral density $J(\omega) = \eta\omega\exp(-\omega/\Omega)$ according to [3]. It can now be shown that $\tilde{a}(t) = \exp(-i\omega t)(A(t)\mathbb{1} + a)$ with $A(t) = \frac{i}{\hbar}\int_0^t dt'\lambda(t')\exp(i\omega t')$. After some algebra we finally arrive at the interaction picture master equation

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\frac{i}{\hbar}[\tilde{w}(t), \tilde{\rho}_S(t)] + \mathcal{D}\left[\tilde{\rho}_S(t)\right], \qquad (3)$$

with the dissipator $\mathcal{D}[\cdot] = \gamma_{12}D_a[\cdot] + \gamma_{21}D_{a^\dagger}[\cdot]$ and $D_O[\cdot] = O\cdot O^\dagger - \{O^\dagger O, \cdot\}/2$ such as $\gamma_{ij} = \int_{\mathbb{R}} ds C_{ij}(s)e^{i\omega_{ij}s}$ with $\omega_{12} = -\omega_{21} = \omega$. Furthermore we have defined $\tilde{w}(t) = i\hbar\left(f^*(t)a - f(t)a^\dagger\right)$ with the driving function

$$f(t) = \int_0^t ds\,(C_{12}(s) - C_{21}^*(s))\,e^{i\omega s}A(t-s). \qquad (4)$$

The master equation (3) is a Redfield master equation that preserves positivity by choice of the interaction Hamiltonian.

Adiabatic master equation

In [1] an adiabatic master equation was rigorously derived for slow driving. The only difference to our master equation is that all instances of $U_S(t)$ and $U_S(t-s)$ are approximated by $U_S^{ad}(t)$ and $e^{isH_S(t)/\hbar}U_S^{ad}(t)$ respectively and with $U_S^{ad}(t)$ being the adiabatic time evolution operator (see [1] for details). For the harmonic oscillator, after some straightforward calculations we obtain the master equation

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\frac{i}{\hbar}[\tilde{w}_{ad}(t), \tilde{\rho}_S(t)] + \mathcal{D}\left[\tilde{\rho}_S(t)\right] \qquad (5)$$

This is exactly our master equation (3) with the only difference that $\tilde{w}(t)$ is replaced by $\tilde{w}_{ad}(t) = i\hbar(f_{ad}^*(t)a - f_{ad}(t)a^\dagger)$ and

$$f_{ad}(t) = \eta\int_0^t ds\frac{e^{i\omega s}}{\left(\frac{1}{\Omega} + is\right)^2}A_{ad}(t, t-s). \qquad (6)$$

where $A_{ad}(t, t-s) = (\lambda(t)/\hbar\omega)\exp(i\omega(t-s))$. In the scope of this work we use a linear driving protocol $\lambda(t) = \hbar\omega\Delta l t/T$, for which one can show that in the limit $T\to\infty$ at constant t/T of slow driving, the non-adiabatic contribution of eq. (3) vanishes. Futhermore we want to compare our master equation with the so-called O'Connell master equation, that can be rigorously derived from eq. (3) in the limit of weak-driving and in the Schrödinger picture is given by

$$\frac{d}{dt}\rho_{OC}(t) = -\frac{i}{\hbar}[H_S(t), \rho_{OC}(t)] + \mathcal{D}\left[\rho_{OC}(t)\right]. \qquad (7)$$

Solution and observables

Analogously to [4], eq. (3) can be solved by the ansatz

$$\tilde{\rho}_S(t) = \exp(\phi(t))\exp(\alpha(t)a^\dagger)\exp(\chi(t)n)\exp(\alpha^*(t)a), \qquad (8)$$

Inserting this into eq. (3) leads to

$$\dot{z}(t) = -2\sigma z(t) + \gamma_{12}z^2(t) + \gamma_{21}, \qquad (9a)$$
$$\dot{\alpha}(t) = (\gamma_{12}z(t) - \sigma)\,\alpha(t) + f(t)(z(t)-1), \qquad (9b)$$

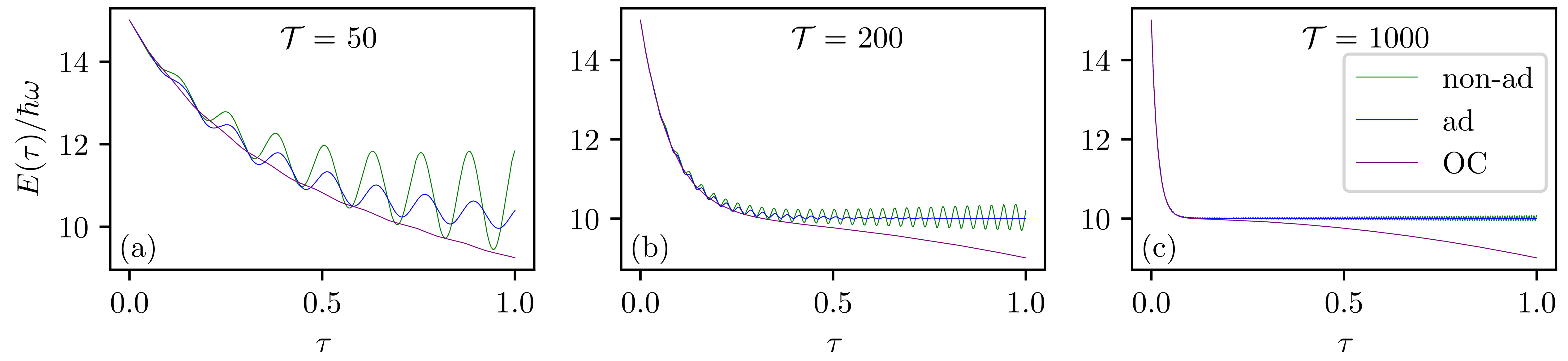
where $\sigma = (\gamma_{12} + \gamma_{21})/2$ and $z(t) = \exp(\chi(t))$. The function $\phi(t)$ can be obtained by normalization. From eq. (8) we can calculate the observables $\langle a\rangle(t) = \langle a^\dagger\rangle^*(t)$ and $\langle n\rangle(t)$ by

$$\langle a\rangle(t) = e^{-i\omega t}\left(A(t) + \alpha(t)\frac{1}{1-z(t)}\right), \qquad (10a)$$

$$\langle n\rangle(t) = |A(t)|^2 + \frac{1}{1-z(t)}\left(z(t) + |\alpha(t)|^2\frac{1}{1-z(t)}\right) + \frac{A^*(t)\alpha(t)}{1-z(t)} + \frac{A(t)\alpha^*(t)}{1-z(t)}. \qquad (10b)$$

Mean energy and coherence

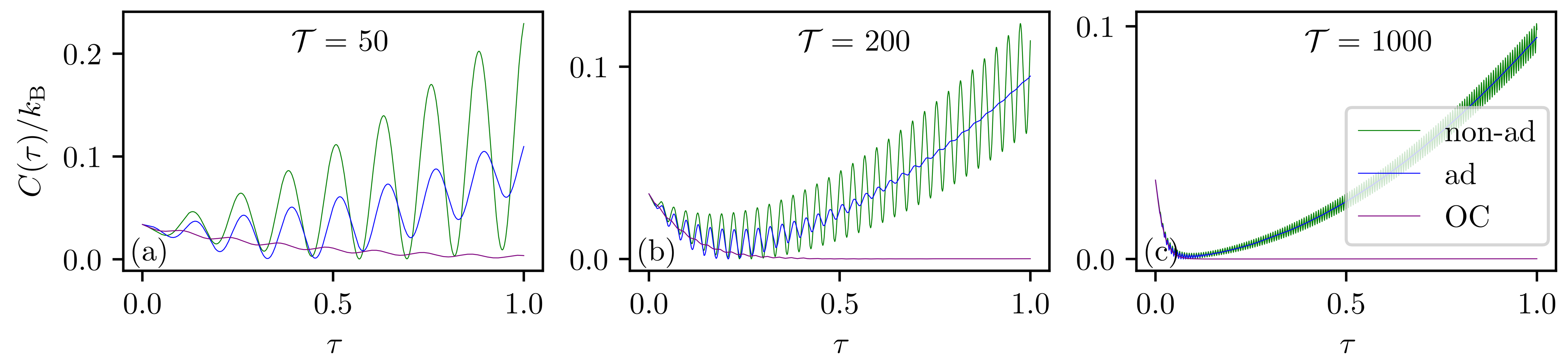
The mean energy of the system can easily be calculated from the observables (10) and is given by $E(t) = \hbar\omega\left(\langle n\rangle(t) + 1/2\right) - \lambda(t)\left(\langle a\rangle(t) + \langle a^\dagger\rangle(t)\right)$ The figure shows the energy in units of $\hbar\omega$ as function of the dimensionless time $\tau = t/T$ with $y = 1/\beta\hbar\omega = 10$, $w = \Omega/\omega = 100$, $\eta = 0.01$, $\Delta l = 1$ and for the three dimensionless driving times $\mathcal{T} = \omega T = 50, 200, 1000$. The initial state is fully determined by $\langle a\rangle(0) = 0.5 + 0.5i$, $\delta n_0 = \langle n\rangle(0) - n_{th} = 5$.



As expected, the adiabatic approximation becomes better with increasing $\mathcal{T}$ and worse with increasing τ. For large enough $\mathcal{T}$ and small τ, the O'Connell master equations approximates the other master equations quite well, while it deviates notably from them for bigger τ. In the quick driving regime all three master equations lie far appart from each other. Another interesting property of the solution of our master equation is the amount of coherences with respect to the instantaneous eigenbasis of $H_S(t)$. In [2], we show that a measure for the latter is given by

$$C_{\mathcal{B}_t}(\rho_S(t)) = k_B\left[\langle n_t\rangle(t) + 1\right]\ln[\langle n_t\rangle(t) + 1] - k_B\langle n_t\rangle(t)\ln[\langle n_t\rangle(t)] - S(\rho_S(t)). \qquad (11)$$

with $n_t = n - \tilde{\lambda}(t)a^\dagger - \tilde{\lambda}(t)a + \tilde{\lambda}^2(t)\mathbb{1}$ being the number operator in the instantaneous eigenbasis $\mathcal{B}(t)$ and $S(\rho) = -k_B\,\mathrm{Tr}(\rho\ln\rho)$ being the von-Neumann entropy. The coherence measure can be interpreted as relative entropy of $\rho_S(t)$ and the incoherent Gaussian state $\delta_{\mathcal{B}_t}(\bar{n}_t)$ with the same expectation value of the number operator n_t corresponding to $\mathcal{B}_t$, i.e. $\bar{n}_t = \mathrm{Tr}(n_t\rho_S(t))$. The figure displays the coherence measure for the same choice of parameters as for the energy.



The intuitive expectation would suggest, that in the adiabatic limit the system would quickly approach the instataneous thermal state, causing the coherence to vanish. This is not the case here. We analyse this in detail in [2].

Conclusion

We derived and solved a Redfield master equation for a driven harmonic oscillator that preserves positivity by the choice of the interaction Hamiltonian. For a linear protocoll we compared it with an adiabatic master derived in [1] and found good agreement in the limit of slow driving in the master equation itsself, in the mean energy and in a coherence measure for Gaussian states. With our work we paved the way to describe the full thermodynamics of an arbitrary quick driven harmonic oscillator weakly coupled to a thermal bath or e.g. a quantum Carnot engine.

References

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