



In this worksheet, we study the density operator of a thermal harmonic oscillator and of a two-level system. Moreover, the papers are concerned with an experimental test of the linearity of quantum mechanics and the realization of Wheeler's delayed-choice experiment.

Exercise 4: Density operator of a thermal harmonic oscillator

Consider a quantum-mechanical harmonic oscillator coupled to a heat bath at temperature T .

- 1) Write down the probability to occupy energy eigenstate $|n\rangle$ and the corresponding density operator. Determine the normalization factor (the partition function).
- 2) Evaluate the mean energy $\langle \hat{H} \rangle$ and determine the average number of quanta $n_{\text{th}} = \langle \hat{n} \rangle$.
- 3) Express the density operator as a function of n_{th} and compute $(\Delta n_{\text{th}})^2 = \langle \hat{n}^2 \rangle - n_{\text{th}}^2$.

Exercise 5: Pair of two-level atoms

Consider two spin-1/2 particles (expressing two two-level atoms). We denote the individual basis states as $|\uparrow\rangle$ and $|\downarrow\rangle$.

- 1) What are the four basis states for the composite system?
- 2) There is one state with spin 0 (singlet) and three states with spin 1 (triplet). Calculate the corresponding density operator. Evaluate the reduced density operator of the first spin, obtained by taking the trace over the second spin.
- 3) Evaluate the purity of each of the above states (with respect to the reduced system). Discuss the physical meaning of the results.

Exercise 6: Density operator of a two-level atom

The simplest and most important model of an atom consists of two discrete and orthogonal states $|b\rangle$ and $|a\rangle$. A complete set of operators used to describe two-level atoms are the four Pauli operators: $1 = |a\rangle\langle a| + |b\rangle\langle b|$, $\sigma_3 = |a\rangle\langle a| - |b\rangle\langle b|$, $\sigma_+ = |a\rangle\langle b|$ and $\sigma_- = |b\rangle\langle a|$ (instead of σ_+ and σ_- the operators $\sigma_1 = \sigma_- + \sigma_+$ and $\sigma_2 = i(\sigma_- - \sigma_+)$ are also used).

- 1) Write the matrix representation of these 6 operators in the basis $|b\rangle, |a\rangle$.
- 2) The density operator of a general state is given by

$$\rho = \rho_{aa} |a\rangle\langle a| + \rho_{ab} |a\rangle\langle b| + \rho_{ba} |b\rangle\langle a| + \rho_{bb} |b\rangle\langle b| .$$

Use 1) to show that $\rho = \frac{1}{2} (1 + u\sigma_1 + v\sigma_2 + w\sigma_3)$ and determine the coefficients u , v and w in terms of the matrix elements of ρ .

- 3) Show that $\rho_{aa} + \rho_{bb} = 1$ and that $\rho_{aa}\rho_{bb} = \rho_{ab}\rho_{ba}$ for pure states. Show that in general $\rho_{aa}\rho_{bb} \geq \rho_{ab}\rho_{ba}$.

Let us now heuristically derive the equation governing the time evolution of ρ for an atom undergoing spontaneous emission.

- 4) The excited state probability ρ_{aa} decays at the Einstein rate $2A$. Use the results of 3) to show that $\rho_{aa} \gtrsim |\rho_{ab}|^2$ assuming that $\rho_{aa}^2 \ll \rho_{aa}$. What does this imply for the decay rate of the matrix element ρ_{ab} ?
- 5) Write down the simplest set of equations for $\dot{\rho}_{ba}$, $\dot{\rho}_{ab}$, $\dot{\rho}_{bb}$ and $\dot{\rho}_{aa}$ consistent with 4). Show that these equations can be combined to the so-called master equation

$$\dot{\rho} = A (2\sigma_- \rho \sigma_+ - \sigma_+ \sigma_- \rho - \rho \sigma_+ \sigma_-) .$$

We shall later see that this is indeed the equation determining the time evolution of ρ .

Exercise 7: Paper-Work

Find the following articles online and answer the following questions for each of them:

- What is the paper about?
- Why is it interesting?
- What is done?
- How is it done?

Test of the linearity of quantum mechanics by RF spectroscopy of the $^9\text{Be}^+$ ground state

J. J. Bollinger, D. J. Heinzen, Wayne M. Itano, S. L. Gilbert, and D. J. Wineland

Phys. Rev. Lett. **63**, 1031 (1989).

Experimental Realization of Wheeler's Delayed-Choice Gedanken Experiment

Vincent Jacques, E Wu, Frédéric Grosshans, François Treussart, Philippe Grangier, Alain Aspect, Jean-François Roch

Science **315**, 966 (2007)