Theoretical quantum optics

Summer semester 2018

Worksheet 3

07.06.2018

This worksheet contains a discussion of the uncertainty relations for number and electric field operators, as well as for number and phase operators. The paper is devoted to the mass of the photon.

Exercise 8: Quantum fluctuations of the single mode electric field

The single mode electric field is given by

$$\hat{E}_x(z,t) = \left(\frac{2\omega^2}{V\epsilon_0}\right)^{\frac{1}{2}} \hat{q}(t)\sin(kz)$$

where $\hat{q}(t)$ is the (canonical) position operator.

1) Calculate the expectation values of $\hat{E}_x(z,t)$ and $\hat{E}_x^2(z,t)$ in the number state $|n\rangle$.
2) Evaluate the fluctuations in the electric field given by $\Delta E_x^2 = \left\langle \hat{E}_x^2 \right\rangle - \left\langle \hat{E}_x \right\rangle^2$. Discuss the dependence on n.
3) Compute the commutator $\left[\hat{n}, \hat{E}_x\right]$ and discuss the physical consequences.
4) Show for any Hermitian operators A and B that $\Delta A \Delta B \geq \frac{1}{2}\left|\langle C\rangle\right|$, where $C = [A, B]$. Use the latter to obtain the uncertainty relation for $\hat{n}$ and $\hat{E}_x$.

Exercise 9: Quantum phase

In classical electrodynamic theory, the electric field of a single mode can be written as:

$$\begin{aligned}\vec{E}\left(\vec{r},t\right) &= E_0 \cos\left(\vec{k}\vec{r} - \omega t + \phi\right)\vec{e}_x \\ &= \frac{E_0}{2}\left[e^{i\left(\vec{k}\vec{r}-\omega t+\phi\right)} + e^{-i\left(\vec{k}\vec{r}-\omega t+\phi\right)}\right]\vec{e}_x \end{aligned} \qquad (1)$$

where E_0 is the amplitude of the field and ϕ its phase. In order to try to quantize Eq. (1), Dirac introduced the following annihilation and creation operators:

$$\hat{a} = e^{i\hat{\phi}}\sqrt{\hat{n}} \qquad\qquad \hat{a}^\dagger = \sqrt{\hat{n}}\, e^{-i\hat{\phi}}$$

where $\hat{n}$ is the number operator and $\hat{\phi}$ is interpreted as a Hermitian operator for the phase.

1) Using the commutation relation $\left[\hat{a}, \hat{a}^\dagger\right] = 1$, show that $e^{i\hat{\phi}}\hat{n}e^{-i\hat{\phi}} - \hat{n} = 1$.
2) By expanding the exponentials, show that the above equation is satisfied as long as $\left[\hat{n}, \hat{\phi}\right] = i$. Derive the corresponding uncertainty relation (compare Ex. 11.4) for the number and phase observables.
3) Consider the matrix elements of the commutator for arbitrary number states $|m\rangle$ and $|n\rangle$, $\langle m|\left[\hat{n}, \hat{\phi}\right]|n\rangle$ for the case $m = n$.
   What does this imply regarding the existence of a Hermitian phase operator?

Exercise 10: Paper-Work

Find the following articles online and answer the following questions for each of them:

- What is the paper about?
- Why is it interesting?
- What is done?
- How is it done?

New Experimental Limit on the Photon Rest Mass with a Rotating Torsion Balance
Jun Luo, Liang-Cheng Tu, Zhong-Kun Hu, and En-Jie Luan
Phys. Rev. Lett. **90**, 081801 (2003).